A Moment Solution for Waveguide Junction Problems

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Abstract —**A moment procedure for solving waveguide junction problems is given using the generalized network formulation for aperture problems. As an illustration, the procedure is applied to a transverse planar junction between two uniform cylindrical waveguides. The generalized admittance network representation of the junction is first obtained. Its scattering matrix representation is then deduced from the former representation. A discussion shows that the conservation of complex power technique, which handles the same class of junctions, is a specialization of the moment procedure.**

I. Introduction

A PROBLEM of practical importance in electromagnetic analysis is the scattering problem at waveguide junctions. This problem is considered solved once an adequate representation of the junction is found. For this reason, a considerable amount of effort has been expended in devising various techniques (both analytical and numerical) to find such representations. The purpose of this paper is to present a moment solution for the waveguide junction problem. The procedure used is based on the generalized network formulation for aperture problems [1]. To illustrate the solution procedure, we apply it to the problem of two infinitely long uniform cylindrical waveguides with a transverse planar junction. Fig. 1 shows the problem at hand. Two different representations of the junction are obtained. The generalized network representation of the junction is first obtained assuming an arbitrary incident field in guide A. The scattering matrix representation of the junction is then deduced from the Galerkin specialization of the generalized network representation. Other moment solutions can be found in the works of Wu and Chow [2] and Chow and Wu [3].

It is the emphasis in this paper to present the moment procedure so that all the results and different relationships are clearly seen. The simple example worked out is particularly chosen to illustrate the relation of the moment procedure to some familiar techniques often used in similar situations as is pointed out in the discussion.

II. The Generalized Network Representation of the Junction

Let the excitation of the junction be a source which produces a multimode field. This source, assumed to be

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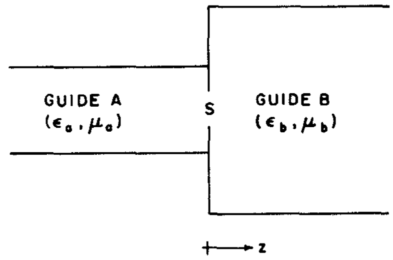

Fig. 1. Two infinitely long uniform cylindrical waveguides opening into each other through an aperture S.

located in guide A, generates outward traveling modes whose amplitudes and phases are not influenced by reflections from the junction. Because of the presence of the junction, part of the incident field is reflected into guide A, while the rest of it is transmitted into guide B. The total field transverse to the z-direction in both guides can be expressed in modal form as [4, sec. 8-2]

$$\boldsymbol{E}_t = \begin{cases} \sum_i c_i e^{-\gamma_{ai} z} \boldsymbol{e}_{ai} + \sum_i a_i e^{\gamma_{ai} z} \boldsymbol{e}_{ai} & (0 > z) \\ \sum_i b_i e^{-\gamma_{bi} z} \boldsymbol{e}_{bi} & (z > 0) \end{cases}$$

$$\boldsymbol{H}_t = \begin{cases} \sum_i c_i Y_{ai} e^{-\gamma_{ai} z} \boldsymbol{u}_z \times \boldsymbol{e}_{ai} & \\ -\sum_i a_i Y_{ai} e^{\gamma_{ai} z} \boldsymbol{u}_z \times \boldsymbol{e}_{ai} & (0 > z) \\ \sum_i b_i Y_{bi} e^{-\gamma_{bi} z} \boldsymbol{u}_z \times \boldsymbol{e}_{bi} & (z > 0). \end{cases} \tag{1}$$

All the modes TE and TM are included in the summation. In (1), c_i, a_i, and b_i are complex coefficients of the ith incident, reflected, and transmitted modes, respectively. γ_{ai} is the modal propagation constant of the ith mode in guide A

$$\gamma_{ai} = \begin{cases} j\beta_i = j\kappa_a \sqrt{1 - \left(\frac{\lambda_a}{\lambda_{ai}}\right)^2} & (\lambda_{ai} > \lambda_a) \\ \alpha_i = \kappa_{ai} \sqrt{1 - \left(\frac{\lambda_{ai}}{\lambda_a}\right)^2} & (\lambda_a > \lambda_{ai}). \end{cases} \tag{2}$$

Here κ_a is the wave number of the medium filling guide A, and κ_{ai} is the ith mode cutoff wavenumber; λ_a and λ_{ai} are the corresponding wavelengths. Y_{ai} is the modal character-

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istic admittance of the ith mode in guide A

$$Y_{ai} = \begin{cases} \dfrac{\gamma_{ai}}{j\omega\mu_a} & \text{(TE modes)} \\ \dfrac{j\omega\epsilon_a}{\gamma_{ai}} & \text{(TM modes)}. \end{cases} \tag{3}$$

The corresponding parameters for guide B are similarly defined. Finally, the modal vectors, $\boldsymbol{e}_{ai}$ in guide A and $\boldsymbol{e}_{bi}$ in guide B, form sets of orthonormal real vectors, viz.,

$$\iint_Q \boldsymbol{e}_{qi} \cdot \boldsymbol{e}_{qj}\, ds = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j). \end{cases} \tag{4}$$

The integration is taken over the cross section of guide $Q(Q \in \{A, B\})$. (In (4), $q \in \{a, b\}$ is understood, but is suppressed hereafter to avoid unnecessary writing.)

As a first step toward deriving the generalized network representation of the junction, the equivalence principle [4, sec. 3-5] is used to divide the problem into two parts. Let the source produce the exciting field when S is covered by a perfect conductor. This field, sometimes referred to as the generator field, is denoted $(\boldsymbol{E}_g, \boldsymbol{H}_g)$. The equivalence principle states that the field in guide A is identical with $(\boldsymbol{E}_g, \boldsymbol{H}_g)$ plus the field produced by the magnetic current sheet

$$\boldsymbol{M} = \boldsymbol{u}_z \times \boldsymbol{E}_t|_{z=0} \tag{5}$$

over S when it is covered by a perfect conductor. The field in guide B is then identical with the field produced by a magnetic current sheet $-\boldsymbol{M}$ over S when it is covered by a perfect conductor. Fig. 2 shows the equivalent situations. The transverse field produced in guide A by $\boldsymbol{M}$, denoted $(\boldsymbol{E}_a(\boldsymbol{M}), \boldsymbol{H}_a(\boldsymbol{M}))$, and that produced in guide B by $-\boldsymbol{M}$, denoted $(\boldsymbol{E}_b(-\boldsymbol{M}), \boldsymbol{H}_b(-\boldsymbol{M}))$, will have the same form as (1), except that there is no exciting field. Hence, the total z-transverse field, equivalent to (1), will be

$$\boldsymbol{E}_t = \begin{cases} \sum_i c_i e^{-\gamma_{ai}z} \boldsymbol{e}_{ai} - \sum_i c_i e^{\gamma_{ai}z} \boldsymbol{e}_{ai} \\ \quad + \sum_i d_i e^{\gamma_{ai}z} \boldsymbol{e}_{ai} & (0 > z) \\ \sum_i b_i e^{-\gamma_{bi}z} \boldsymbol{e}_{bi} & (z > 0) \end{cases}$$

$$\boldsymbol{H}_t = \begin{cases} \sum_i c_i Y_{ai} e^{-\gamma_{ai}z} \boldsymbol{u}_z \times \boldsymbol{e}_{ai} + \sum_i c_i Y_{ai} e^{\gamma_{ai}z} \boldsymbol{u}_z \times \boldsymbol{e}_{ai} \\ \quad - \sum_i d_i Y_{ai} e^{\gamma_{ai}z} \boldsymbol{u}_z \times \boldsymbol{e}_{ai} & (0 > z) \\ \sum_i b_i Y_{bi} e^{-\gamma_{bi}z} \boldsymbol{u}_z \times \boldsymbol{e}_{bi} & (z > 0). \end{cases} \tag{6}$$

Here c_i, d_i, and b_i are the respective coefficients of the ith incident mode, the ith mode produced by $\boldsymbol{M}$, and the ith mode produced by $-\boldsymbol{M}$. $\boldsymbol{M}$ can be evaluated from (5) and

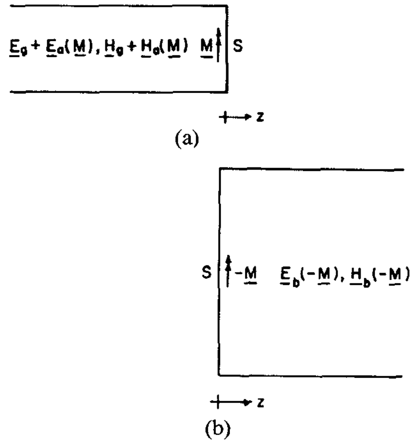

Fig. 2. (a) The equivalent situation for guide A. $\boldsymbol{M}$ exists only on S. (b) The equivalent situation for guide B. $-\boldsymbol{M}$ exists only on S.

(6) as

$$\boldsymbol{M} = \boldsymbol{u}_z \times \boldsymbol{E}_a(\boldsymbol{M})|_{z=0} = \sum_i d_i \boldsymbol{u}_z \times \boldsymbol{e}_{ai}$$

and

$$\boldsymbol{M} = \boldsymbol{u}_z \times \boldsymbol{E}_b(-\boldsymbol{M})|_{z=0} = \sum_i b_i \boldsymbol{u}_z \times \boldsymbol{e}_{bi}, \qquad \text{on } S. \tag{7}$$

The placement of magnetic current sheets $+\boldsymbol{M}$ over S in guide A and $-\boldsymbol{M}$ over S in guide B ensures the continuity of $\boldsymbol{E}_t$ across S. The continuity of $\boldsymbol{H}_t$ across S, however, requires that

$$2\sum_i c_i Y_{ai} \boldsymbol{u}_z \times \boldsymbol{e}_{ai} = \sum_i d_i Y_{ai} \boldsymbol{u}_z \times \boldsymbol{e}_{ai} + \sum_i b_i Y_{bi} \boldsymbol{u}_z \times \boldsymbol{e}_{bi}, \qquad \text{on } S. \tag{8}$$

If (8) were satisfied exactly, we would have the true solution. To obtain an approximate solution, we apply the method of moments [5]. This is the second and final step toward deriving the generalized network representation of the junction.

Let $\{\boldsymbol{M}_j\}$, $j = 1, 2, \cdots, N$, be a set of real-valued expansion functions, and put

$$\boldsymbol{M} = \sum_{j=1}^{N} V_j \boldsymbol{M}_j \tag{9}$$

where V_j are complex coefficients to be determined. Since the set $\{\boldsymbol{u}_z \times \boldsymbol{e}_{qj}\}$, $j = 1, 2, \cdots$, is complete [6, sec. 5.6], a finite subset of the lower order modes can be used in (8) to approximate the $\boldsymbol{H}_t$ field in guide Q $(Q \in \{A, B\})$. Henceforth, the number of modes in the modal expansion is assumed to be L. (The results obtained would, nevertheless, hold, had the number of modes in guide A been different from that in guide B.) Substituting (9) into (7), we obtain

$$\sum_{j=1}^{N} V_j \boldsymbol{M}_j = \sum_{i=1}^{L} d_i \boldsymbol{u}_z \times \boldsymbol{e}_{ai}$$

and

$$\sum_{j=1}^{N} V_j \boldsymbol{M}_j = \sum_{i=1}^{L} b_i \boldsymbol{u}_z \times \boldsymbol{e}_{bi}. \tag{10}$$

Scalarly multiplying the first equation of (10) by $\boldsymbol{u}_z \times \boldsymbol{e}_{a\kappa}$, and the second by $\boldsymbol{u}_z \times \boldsymbol{e}_{b\kappa}$, $\kappa = 1,2,\cdots,L$, and integrating over the corresponding guide cross sections, we obtain

$$\sum_{j=1}^{N} V_j \iint_A \boldsymbol{M}_j \cdot \boldsymbol{u}_z \times \boldsymbol{e}_{a\kappa}\, ds = \sum_{i=1}^{L} d_i \iint_A \boldsymbol{u}_z \times \boldsymbol{e}_{a\kappa} \cdot \boldsymbol{u}_z \times \boldsymbol{e}_{ai}\, ds$$

$$\sum_{j=1}^{N} V_j \iint_B \boldsymbol{M}_j \cdot \boldsymbol{u}_z \times \boldsymbol{e}_{b\kappa}\, ds = \sum_{i=1}^{L} b_i \iint_B \boldsymbol{u}_z \times \boldsymbol{e}_{b\kappa} \cdot \boldsymbol{u}_z \times \boldsymbol{e}_{bi}\, ds.$$

Because of the mode orthogonality relationship (4), all the terms in the summation on the right vanish except the $i = \kappa$ term. Hence

$$\left.\begin{aligned} d_i &= \sum_{j=1}^{N} V_j H_{aij} \\ b_i &= \sum_{j=1}^{N} V_j H_{bij} \end{aligned}\right\} \quad (i = 1,2,\cdots,L) \tag{11}$$

where

$$H_{qij} = \iint_S \boldsymbol{M}_j \cdot \boldsymbol{u}_z \times \boldsymbol{e}_{qi}\, ds \qquad (Q \in \{A,B\}). \tag{12}$$

The integrals over the different guide cross sections in (12) are replaced by one over S since $\boldsymbol{M}_j$ exists only in this region. Next, define a symmetric product

$$\langle \boldsymbol{F}, \boldsymbol{G} \rangle = \iint_S \boldsymbol{F} \cdot \boldsymbol{G}\, ds \tag{13}$$

and a set of real-valued testing functions $\{\boldsymbol{W}_j\}$, $j = 1,2,\cdots,N$. Taking the symmetric product of (8) with each testing function $\boldsymbol{W}_\kappa$, and using (11), we obtain the set of equations

$$2\sum_{i=1}^{L} c_i Y_{ai} W_{ai\kappa} = \sum_{i=1}^{L}\left(\sum_{j=1}^{N} V_j H_{aij}\right) Y_{ai} W_{ai\kappa} + \sum_{i=1}^{L}\left(\sum_{j=1}^{N} V_j H_{bij}\right) Y_{bi} W_{bi\kappa} \qquad (\kappa = 1,2,\cdots,N) \tag{14}$$

where

$$W_{qi\kappa} = \iint_S \boldsymbol{W}_\kappa \cdot \boldsymbol{u}_z \times \boldsymbol{e}_{qi}\, ds \qquad (Q \in \{A,B\}). \tag{15}$$

This set of equations can be put in matrix form as follows. Define the mode-coefficient vectors

$$\vec{c} = [c_i]_{L\times 1} \tag{16}$$

$$\vec{d} = [d_i]_{L\times 1} = H_a \vec{V} \tag{17}$$

and

$$\vec{b} = [b_i]_{L\times 1} = H_b \vec{V} \tag{18}$$

where (17) and (18) follow from (11). Here $\vec{V}$ is the coefficient vector

$$\vec{V} = [V_i]_{N\times 1} \tag{19}$$

and H_a and H_b are matrices given by

$$H_q = [H_{qij}]_{L\times N} = \left[\iint_S \boldsymbol{M}_j \cdot \boldsymbol{u}_z \times \boldsymbol{e}_{qi}\, ds\right] \qquad (Q \in \{A,B\}). \tag{20}$$

The system of equations (14) will then have the form

$$[\overline{Y}_a + \overline{Y}_b]\vec{V} = \vec{I} \tag{21}$$

where the generalized admittance matrices, $\overline{Y}_a$ of guide A and $\overline{Y}_b$ of guide B, are given by

$$\overline{Y}_q = [\overline{Y}_{qij}]_{N\times N} = W_q^T Y_q H_q \qquad (Q \in \{A,B\}) \tag{22}$$

and the source vector $\vec{I}$ is given by

$$\vec{I} = [I_i]_{N\times 1} = 2W_a^T Y_a \vec{c}. \tag{23}$$

In (22) and (23), W_q is exactly the same as H_q except for $\boldsymbol{W}_j$ replacing $\boldsymbol{M}_j$, Y_q is a diagonal matrix of the modal characteristic admittances of guide Q ($Q \in \{A,B\}$), and T denotes matrix transpose.

By (21) we have finally arrived at the generalized network representation of the junction. Equation (21) can be interpreted as two generalized networks $\overline{Y}_a$ and $\overline{Y}_b$ in parallel with the current source $\vec{I}$, a situation shown in Fig. 3. The junction can also be completely described by its scattering matrix representation. This representation can be deduced almost immediately from the generalized network representation, as will be seen shortly. First, however, we prove that the continuity of complex power flow across the junction is preserved under its generalized network representation specialized to the Galerkin case. The proof essentially follows the outline given by Mautz and Harrington [7].

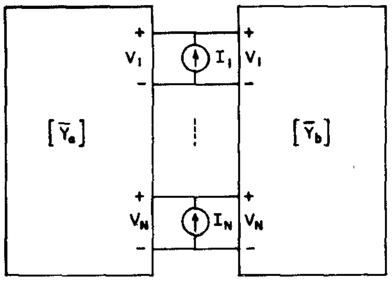

Fig. 3. The generalized network interpretation of (21).

III. Continuity of Complex Power Flow Across the Junction

The continuity of complex power flow across the junction requires that the total complex power on both sides of the junction be equal. The complex power transmitted through the junction into guide B is basically

$$P_t = \iint_S \boldsymbol{E} \times \boldsymbol{H}^* \cdot \boldsymbol{u}_z\, ds \tag{24}$$

where the asterisk denotes complex conjugate. Substituting from (5), (24) becomes

$$P_t = \iint_S \boldsymbol{M} \cdot \boldsymbol{H}^* \, ds. \tag{25}$$

Equation (25) involves only the tangential component of $\boldsymbol{H}$ over S in guide B. Thus, on substituting from (6) and using (9), we get

$$P_t = \sum_{i=1}^{N} V_i \sum_{j=1}^{L} Y_{bj}^* b_j^* \left(\iint_S \boldsymbol{M}_i \cdot \boldsymbol{u}_z \times \boldsymbol{e}_{bj} \, ds \right)$$

which can be put in matrix form as

$$P_t = \vec{V}^H H_b^T Y_b^* H_b \vec{V}. \tag{26}$$

In (26), the superscript H denotes conjugate transpose, and (18) was used to replace $\vec{b}$ by $H_b\vec{V}$. The total complex power entering the junction from guide A can be expressed in the form (25). In this case, the integral involves the tangential component of $\boldsymbol{H}$ over S in guide A. Thus, on substituting from (6) and using (9), we get

$$P_{\text{in}} = \sum_{i=1}^{N} V_i \sum_{j=1}^{L} Y_{aj}^* (2c_j^* - d_j^*) \iint_S \boldsymbol{M}_i \cdot \boldsymbol{u}_z \times \boldsymbol{e}_{aj} \, ds$$

which can be written in matrix form as

$$P_{\text{in}} = 2\vec{c}^H Y_a^* H_a \vec{V} - \vec{V}^H H_a^T Y_a^* H_a \vec{V}. \tag{27}$$

Here (17) was used to replace $\vec{d}$ by $H_a\vec{V}$. For the power flow across the junction to be continuous, P_t must be equal to P_{in}. Consider the Galerkin case where $\{\boldsymbol{M}_j\} = \{\boldsymbol{W}_j\}$, $j = 1, 2, \cdots, N$. Then

$$H_q = W_q \qquad (Q \in \{A, B\})$$

and on using (22) and (23), (26) and (27) become

$$P_t = \vec{V}^H \overline{Y}_b^H \vec{V} \tag{28}$$

and

$$P_{\text{in}} = \vec{I}^H \vec{V} - \vec{V}^H \overline{Y}_a^H \vec{V}. \tag{29}$$

The equality of P_t and P_{in} is therefore guaranteed because of (21). Since a finite number of modes is used in expanding the field in guides A and B, P_{in} and P_t, as given respectively by (29) and (28), are only approximations to the power on both sides of the junction. Because of mode completeness, as the number of modes tends to infinity, P_{in} and P_t converge to the actual powers. The continuity of power flow across the junction then becomes exact.

IV. The Scattering Matrix Representation of the Junction

Following Montgomery *et al.* [8, sec. 5-14], we define the scattering matrix of the junction of two waveguides A and B to be the matrix

$$S = \begin{bmatrix} S_{aa} & S_{ab} \\ S_{ba} & S_{bb} \end{bmatrix} \tag{30}$$

where the ijth element of S_{qo} gives the amplitude of the ith mode in guide Q due to the jth incident mode of unit amplitude in guide Q ($\{Q, O\} = \{A, B\}$).

The scattering submatrices S_{aa} and S_{ba} can be extracted from the analysis in Section II specialized to the Galerkin case. Let $\vec{a}$ be the coefficient vector of the reflected modes, viz.,

$$\vec{a} = [a_i]_{L \times 1}. \tag{31}$$

We have, by (1) and (6)

$$\vec{a} = \vec{d} - \vec{c}$$

and by (17), (21), and (23)

$$\vec{a} = \left(2H_a[\overline{Y}_a + \overline{Y}_b]^{-1} H_a^T Y_a - U\right)\vec{c}.$$

Here U is the identity matrix. It also follows from (18), (21), and (23) that

$$\vec{b} = \left(2H_b[\overline{Y}_a + \overline{Y}_b]^{-1} H_a^T Y_a\right)\vec{c}.$$

Thus, the scattering submatrices S_{aa} and S_{ba} are given by

$$S_{aa} = 2H_a[\overline{Y}_a + \overline{Y}_b]^{-1} H_a^T Y_a - U \tag{32}$$

and

$$S_{ba} = 2H_b[\overline{Y}_a + \overline{Y}_b]^{-1} H_a^T Y_a. \tag{33}$$

As a matter of convenience, (33) is rewritten to read

$$S_{ba} = H(S_{aa} + U) \tag{34}$$

where H is an L by L matrix satisfying

$$H_b = HH_a \tag{35}$$

or, upon multiplying from the right by $\vec{V}$ and using (17) and (18)

$$\vec{b} = H\vec{d}. \tag{36}$$

The matrix H can be evaluated as follows. Scalarly multiply the second equation of (7) by $\boldsymbol{u}_z \times \boldsymbol{e}_{bj}$, $j = 1, 2, \cdots, L$, and integrate over the cross section of guide B to get

$$\iint_S \boldsymbol{M} \cdot \boldsymbol{u}_z \times \boldsymbol{e}_{bj} \, ds = b_j \qquad (j = 1, 2, \cdots, L).$$

Using the first equation of (7) to substitute for $\boldsymbol{M}$ above, we get

$$\sum_{i=1}^{L} d_i \left(\iint_S \boldsymbol{u}_z \times \boldsymbol{e}_{ai} \cdot \boldsymbol{u}_z \times \boldsymbol{e}_{bj} \, ds \right) = b_j \qquad (j = 1, 2, \cdots, L). \tag{37}$$

H is therefore the matrix

$$H = [H_{ij}]_{L \times L} = \left[\iint_S \boldsymbol{u}_z \times \boldsymbol{e}_{bi} \cdot \boldsymbol{u}_z \times \boldsymbol{e}_{aj} \, ds \right]. \tag{38}$$

It can readily be shown in a similar way that

$$\vec{d} = H^T \vec{b}. \tag{39}$$

The scattering submatrices S_{ab} and S_{bb} are due to an incident field in guide B. This situation is reciprocal to the one in Section II. Since H_a, H_b, W_a, and W_b depend only on the functional form of the different expansion and testing functions, keeping these functions unchanged, we get

$$\vec{a}_r + \vec{c}_r = H_b \vec{V}_r \tag{40}$$

$$\vec{b}_r = H_a \vec{V}_r \tag{41}$$

$$[\overline{Y}_a + \overline{Y}_b]\vec{V}_r = \vec{I}_r \tag{42}$$

and

$$\vec{I}_r = 2W_b^T Y_b \vec{c}_r \tag{43}$$

in analogy with (17), (18), (21), and (23), respectively. The vectors $\vec{a}_r$, $\vec{c}_r$, $\vec{b}_r$, $\vec{I}_r$, and $\vec{V}_r$ in the reciprocal case bear the same meanings as do their counterparts $\vec{a}$, $\vec{c}$, $\vec{b}$, $\vec{I}$, and $\vec{V}$. Specializing to the Galerkin case, it immediately follows from (30), (41), (42), and (43) that

$$S_{ab} = 2H_a[\bar{Y}_a + \bar{Y}_b]^{-1} H_b^T Y_b \tag{44}$$

or, upon using (33)

$$S_{ab} = Y_a^{-1} S_{ba}^T Y_b. \tag{45}$$

Finally, by (30), (35), (40), and (41)

$$(S_{bb} + U)\vec{c}_r = HS_{ab}\vec{c}_r.$$

Thus

$$S_{bb} = HS_{ab} - U. \tag{46}$$

V. An Example

Consider the system of two waveguides shown in Fig. 1, with the aperture S being the whole cross section of guide A, and let $\{M_j\} = \{W_j\} = \{u_z \times e_{aj}\}, j = 1, 2, \cdots, N$ $(N = L)$. Then

$$H_a = W_a = U \tag{47}$$

$$H_b = W_b = H \tag{48}$$

$$\bar{Y}_a = Y_a \tag{49}$$

$$\bar{Y}_b = H^T Y_b H \tag{50}$$

and

$$\vec{I} = 2Y_a \vec{c}. \tag{51}$$

The scattering submatrix S_{aa} is then given by

$$S_{aa} = \left(Y_a + H^T Y_b H\right)^{-1}\left(Y_a - H^T Y_b H\right). \tag{52}$$

The other submatrices are given by (34), (45), and (46). Setting $\{e_{aj}\} = \{e_{bj}\}, j = 1, 2, \cdots, N$, it is an easy matter to show that S_{aa}, S_{ba}, S_{ab}, and S_{bb} are given by

$$S_{aa} = (Y_a + Y_b)^{-1}(Y_a - Y_b) \tag{53}$$

$$S_{ba} = 2(Y_a + Y_b)^{-1} Y_a \tag{54}$$

$$S_{ab} = 2(Y_a + Y_b)^{-1} Y_b \tag{55}$$

and

$$S_{bb} = -S_{aa} = (Y_a + Y_b)^{-1}(Y_b - Y_a) \tag{56}$$

a result which should have been expected.

VI. Discussion

A moment solution for waveguide junction problems is given in this paper. The procedure, based on the generalized network formulation for aperture problems, is applied to a transverse planar junction between two uniform cylindrical waveguides. It is clear from the analysis in Section II that a judicious choice of the expansion functions for the equivalent magnetic current is a key to the success of the procedure. For some configurations, such as the one considered in Section V, the choice is quite obvious. For junctions with arbitrarily shaped apertures, triangular patches with appropriate functions defined on each triangle may be used to closely approximate the current on the aperture. Junctions with arbitrary apertures can therefore be treated in a systematic manner. Furthermore, the procedure can be readily extended to handle junctions with more than one aperture. Another important feature of the moment procedure is that the scattering submatrices can be expressed in terms of matrices each of which depends on the modes of only one waveguide. Thus, for a given junction, different waveguides can be considered one at a time. The scattering submatrices are then obtained for any required combination, which adds another measure of flexibility. Computer codes that take into consideration these points are now under preparation.

The moment procedure is straightforward and rather general. In the case of the example in Section V, with the choice of the expansion and testing functions there, it reduces to the familiar mode-matching technique. In a recent publication [9], Safavi-Naini and MacPhie deter mined the scattering matrix for the junction configuration of the example in Section V by employing the principles of conservation of complex power and mode matching across the junction. Apart from multiplying factors due to the mode normalization used in the moment procedure, the scattering matrices obtained are identical. In Section III, the principle of conservation of complex power across the junction was seen to be preserved under its generalized network representation. As a matter of fact, the complex power technique can be regarded as a specialization of the moment procedure, and is probably better viewed in this context.

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Dielectrically Loaded Corrugated Waveguide: Variational Analysis of a Nonstandard Eigenproblem

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Abstract —Motivated by simple fabricability, the dielectrically loaded corrugated waveguide is analyzed applying the theory of nonstandard eigenvalues and variational principles recently presented by one of the authors. The eigenvalue parameter of this problem is the boundary susceptance of the corrugated surface, which choice is seen to lead to a simple functional. The functional is tested for the air-filled corrugated guide, and good accuracy for simple test functions is observed. Dispersion relation for the loaded corrugated guide is calculated together with the field pattern for quasi-balanced operation and estimates for the dielectric loss. The method presented here also appears to be applicable in other waveguide problems where inhomogeneous material is involved.

I. Introduction

THE CORRUGATED WAVEGUIDE has proven useful for different slow-wave structure applications and for radiating systems requiring rotational symmetry of the power radiation pattern [1]. One of the drawbacks of the corrugated structure is its tedious and costly fabrication. A new method of fabrication was, however, recently suggested by Tiuri,[1] which is quite simple: A dielectric rod is put in a lathe, thin grooves are made on the outside, and the outer surface is metallized. To reduce losses, a hole can be drilled on the axis and we have a dielectrically loaded corrugated waveguide. We are concerned here about the analysis of such a structure.

The conventional air-filled corrugated waveguide can be conveniently analyzed in terms of special functions for the circular cylindrical geometry. The additional dielectric interface, however, makes this approach very complicated. So, a variational method is attempted instead. The eigenvalue problem, however, is not of the standard from $Lf = \lambda Mf$, $Bf = 0$, but of the more general form $L(\lambda)f = 0$, $B(\lambda)f = 0$, i.e., the eigenvalue parameter λ does not appear in the differential equation system in linear form, and it might also be present in the boundary conditions. This more general form of an eigenvalue problem was called a nonstandard eigenvalue problem in recent studies [2], [3], where a variational principle for such problems was also formulated. This method will be applied here. The eigenvalue parameter may be chosen freely among all the parameters of the problem. A stationary functional results if the following functional equation can be solved for the eigenvalue parameter λ:

$$(f, L(\lambda)f) + (f, B(\lambda)f)_b = 0 \qquad (1)$$

where the inner products $(\cdot,\cdot), (\cdot,\cdot)_b$ are defined in the domains of the operators L and B, respectively.

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